1.4.1 Hyperbolic Functions

Certain even and odd combination of exponential functions e^x and e^{-x} arise so frequently in mathematics and its applications that they deserve to be given special names. In many ways they are analogous to the trigonometric functions, and they have the same relationship to the hyperbola that the trigonometric functions have to the circle. For this reason they are collectively called **hyperbolic functions** and individually called **hyperbolic sine, hyperbolic cosine** and so on.

DEFINITION OF THE HYPERBOLIC FUNCTIONS

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

The graphs of hyperbolic sine and cosine can be sketched using graphical addition as in Figures 1 and 2.

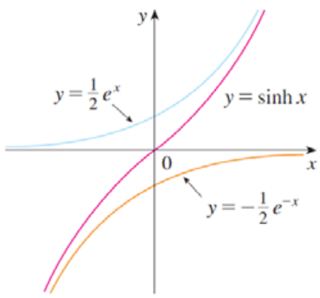

FIGURE 1
$y=\sinh x=\frac{1}{2}e^x-\frac{1}{2}e^{-x}$

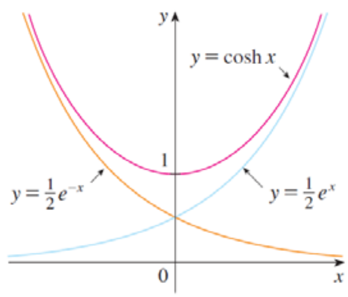

FIGURE 2
$y=\cosh x=\frac{1}{2}e^x+\frac{1}{2}e^{-x}$

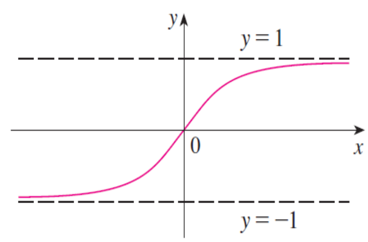

FIGURE 3
$y=\tanh x$

Note that $\sinh(x)$ has domain R and range R, while $\cosh(x)$ has domain R and range $(1,\infty)$. The graph of $\tanh(x)$ is shown in Figure 3. It has the horizontal asymptotes $y=\pm 1$. (See Exercise 23.)

Some of the mathematical uses of hyperbolic functions will be seen in Chapter 7. Applications to science and engineering occur whenever an entity such as light, velocity, electricity, or radioactivity is gradually absorbed or extinguished, for the decay can be represented by hyperbolic functions. The most famous application is the use of hyperbolic cosine to describe the shape of a hanging wire. It can be proved that if a heavy flexible cable (such as a telephone or power line) is suspended between two points at the same height, then it takes the shape of a curve with equation y = c + a cosh(x/a) called a **catenary** (see Figure 4). (the Latin word *catenameans* "chain.")

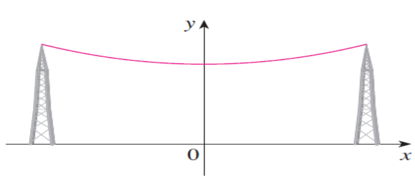

FIGURE 4
A catenary $y=c+a\cosh(x/a)$

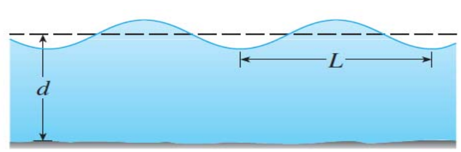

FIGURE 5
Idealized ocean wave

Another application of hyperbolic functions occurs in the description of ocean waves. The velocity of a water wave with length L moving across a body of water with depth d is modeled by the function

$$v=\sqrt{\frac{gL}{2\pi}\tanh\left(\frac{2\pi d}{L}\right)}$$

where g is the acceleration due to gravity. (See Figure 5 and Exercise 49.)

The hyperbolic functions satisfy a number of identities that are similar to well-known trigonometric identities. We list some of them here and leave most of the proofs to the exercises.

HYPERBOLIC IDENTITIES

$\sinh(-x) = -\sinh x \qquad \cosh(-x) = \cosh x$

$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$

$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

EXAMPLE 1 : Prove **(a)** $\cosh^2(x)-\sinh^2(x)=1$ and **(b)** $1-\tanh^2(x)=\operatorname{sech}^2(x)$

$$(a)\quad \cosh^2(x)-\sinh^2(x)=\left(\frac{e^x+e^{-x}}{2}\right)^2-\left(\frac{e^x-e^{-x}}{2}\right)^2$$

$$=\frac{e^{2x}+2+e^{-2x}}{4}-\frac{e^{2x}-2+e^{-2x}}{4}=\frac{4}{4}=1$$

(b) We start with the identity proved in part **(a):** $\cosh^2(x)-\sinh^2(x)=1$

If we divide both sides by $\cosh(x)$, we get

$$1-\frac{\sinh^2(x)}{\cosh^2(x)}=\frac{1}{\cosh^2(x)}$$

$$or \qquad 1-\tanh^2(x)=\sec h^2(x)$$

The identity proved in example 1(a) gives a clue to the reason for the name “hyperbolic” functions. If t is any real number then the point $P(\cos t,\sin t)$ lies on the unit circle $x^2+y^2=1$ because $\cos^2(x)+\sin^2(x)=1$. In fact,, t can be interpreted as the radian measure of $\angle POQ$ in Figure 6. For this reason the trigonometric functions are sometimes called *circular* functions. Likewise, if t is any real number, then the point $P(\cosh t,\sinh t)$ lies on the right branch of the hyperbola $x^2-y^2=1$ because $\cosh^2(x)-\sinh^2(x)=1$ and $\cosh(x)\geq 1$. This time, t does not represent the measure of an angle. However, it turns out that t represents twice the area of the shaded hyperbolic sector in Figure 7, just as in the trigonometric case t represents twice the area of the shaded circular section in Figure 6.

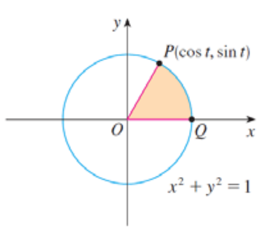

FIGURE 6

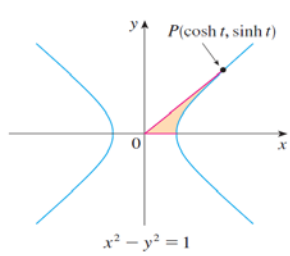

FIGURE 7

Exercises:

1–6 Find the numerical value of each expression.

1. (a) $\sinh 0$ (b) $\cosh 0$

2. (a) $\tanh 0$ (b) $\tanh 1$

3. (a) $\sinh(\ln 2)$ (b) $\sinh 2$

4. (a) $\cosh 3$ (b) $\cosh(\ln 3)$

5. (a) $\operatorname{sech} 0$ (b) $\cosh^{-1} 1$

6. (a) $\sinh 1$ (b) $\sinh^{-1} 1$

7–19 Prove the identity.

7. $\sinh(-x) = -\sinh x$
(This shows that sinh is an odd function.)

8. $\cosh(-x) = \cosh x$
(This shows that cosh is an even function.)

9. $\cosh x + \sinh x = e^x$

10. $\cosh x - \sinh x = e^{-x}$

11. $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

12. $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

13. $\coth^2 x - 1 = \operatorname{csch}^2 x$

14. $\tanh(x + y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

15. $\sinh 2x = 2 \sinh x \cosh x$

16. $\cosh 2x = \cosh^2 x + \sinh^2 x$

17. $\tanh(\ln x) = \dfrac{x^2 - 1}{x^2 + 1}$

18. $\dfrac{1 + \tanh x}{1 - \tanh x} = e^{2x}$

19. $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$
(n any real number)

20. If $\tanh x = \frac{12}{13}$, find the values of the other hyperbolic functions at x.

21. If $\cosh x = \frac{5}{3}$ and $x > 0$, find the values of the other hyperbolic functions at x.

22. (a) Use the graphs of sinh, cosh, and tanh in Figures 1–3 to draw the graphs of csch, sech, and coth.

(b) Check the graphs that you sketched in part (a) by using a graphing device to produce them.

Answers:

1. (a) 0 (b) 1 **3.** (a) $\frac{3}{4}$ (b) $\frac{1}{2}(e^2 - e^{-2}) \approx 3.62686$

5. (a) 1 (b) 0

21. $\operatorname{sech} x = \frac{3}{5}$, $\sinh x = \frac{4}{3}$, $\operatorname{csch} x = \frac{3}{4}$, $\tanh x = \frac{4}{5}$, $\coth x = \frac{5}{4}$