Review of Convergence Tests

NAME	STATEMENT	COMMENTS
Divergence Test	If $\lim_{k\to+\infty} u_k \neq 0$, then Σu_k diverges.	If $\lim_{k\to+\infty} u_k = 0$, Σu_k may or may not converge.
Integral Test	Let Σu_k be a series with positive terms, and let $f(x)$ be the function that results when k is replaced by x in the formula for u_k. If f is decreasing and continuous for $x \geq 1$, then $\sum_{k=1}^{\infty} u_k$ and $\int_1^{+\infty} f(x)\,dx$ both converge or both diverge.	Use this test when $f(x)$ is easy to integrate. This test only applies to series that have positive terms.
Comparison Test	Let Σa_k and Σb_k be series with nonnegative terms such that $a_1 \leq b_1,\ a_2 \leq b_2, \ldots, a_k \leq b_k, \ldots$ If Σb_k converges, then Σa_k converges, and if Σa_k diverges, then Σb_k diverges.	Use this test as a last resort; other tests are often easier to apply. This test only applies to series with nonnegative terms.
Ratio Test	Let Σu_k be a series with positive terms and suppose $\lim_{k\to+\infty} \frac{u_{k+1}}{u_k} = \rho$ (a) Series converges if $\rho < 1$. (b) Series diverges if $\rho > 1$ or $\rho = +\infty$. (c) No conclusion if $\rho = 1$.	Try this test when u_k involves factorials or kth powers.
Root Test	Let Σu_k be a series with positive terms such that $\rho = \lim_{k\to+\infty} \sqrt[k]{u_k}$ (a) Series converges if $\rho < 1$. (b) Series diverges if $\rho > 1$ or $\rho = +\infty$. (c) No conclusion if $\rho = 1$.	Try this test when u_k involves kth powers.
Limit Comparison Test	Let Σa_k and Σb_k be series with positive terms such that $\rho = \lim_{k\to+\infty} \frac{a_k}{b_k}$ If $0 < \rho < +\infty$, then both series converge or both diverge.	This is easier to apply than the comparison test, but still requires some skill in choosing the series Σb_k for comparison.
Alternating Series Test	The series $a_1 - a_2 + a_3 - a_4 + \cdots$ $-a_1 + a_2 - a_3 + a_4 - \cdots$ converge if (a) $a_1 > a_2 > a_3 > \cdots$ (b) $\lim_{k\to+\infty} a_k = 0$	This test applies only to alternating series. It is assumed that $a_k > 0$ for all k.

Summary of convergence and divergence tests for series

TEST	SERIES	CONVERGENCE OR DIVERGENCE	COMMENTS
nth-term	$\sum a_n$	Diverges if $\lim_{n\to\infty} a_n \neq 0$	Inconclusive if $\lim_{n\to\infty} a_n = 0$
Geometric series	$\sum_{n=1}^{\infty} ar^{n-1}$	(i) Converges with sum $S = \dfrac{a}{1-r}$ if $\lvert r\rvert < 1$ (ii) Diverges if $\lvert r\rvert \geq 1$	Useful for comparison tests if the nth term a_n of a series is *similar* to ar^{n-1}
p-series	$\sum_{n=1}^{\infty} \dfrac{1}{n^p}$	(i) Converges if $p > 1$ (ii) Diverges if $p \leq 1$	Useful for comparison tests if the nth term a_n of a series is *similar* to $1/n^p$
Integral	$\sum_{n=1}^{\infty} a_n$ $a_n = f(n)$	(i) Converges if $\int_1^{\infty} f(x)\,dx$ converges (ii) Diverges if $\int_1^{\infty} f(x)\,dx$ diverges	The function f obtained from $a_n = f(n)$ must be continuous, positive, decreasing, and readily integrable.
Comparison	$\sum a_n, \sum b_n$ $a_n > 0, b_n > 0$	(i) If $\sum b_n$ converges and $a_n \leq b_n$ for every n, then $\sum a_n$ converges. (ii) If $\sum b_n$ diverges and $a_n \geq b_n$ for every n, then $\sum a_n$ diverges. (iii) If $\lim_{n\to\infty} (a_n/b_n) = c > 0$, then both series converge or both diverge.	The comparison series $\sum b_n$ is often a geometric series or a p-series. To find b_n in (iii), consider only the terms of a_n that have the greatest effect on the magnitude.
Ratio	$\sum a_n$	If $\lim\limits_{n\to\infty} \left\lvert \dfrac{a_{n+1}}{a_n} \right\rvert = L$ (or ∞), the series (i) converges (absolutely) if $L < 1$ (ii) diverges if $L > 1$ (or ∞)	Inconclusive if $L = 1$ Useful if a_n involves factorials or nth powers If $a_n > 0$ for every n, the absolute value sign may be disregarded.
Root	$\sum a_n$	If $\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} = L$ (or ∞), the series (i) converges (absolutely) if $L < 1$ (ii) diverges if $L > 1$ (or ∞)	Inconclusive if $L = 1$ Useful if a_n involves nth powers If $a_n > 0$ for every n, the absolute value sign may be disregarded.
Alternating series	$\sum (-1)^n a_n$ $a_n > 0$	Converges if $a_k \geq a_{k+1}$ for every k and $\lim_{n\to\infty} a_n = 0$	Applicable only to an alternating series
$\sum \lvert a_n\rvert$	$\sum a_n$	If $\sum \lvert a_n\rvert$ converges, then $\sum a_n$ converges.	Useful for series that contain both positive and negative terms